1 - 7 General solution
Solve the following ODEs, showing the details of your work.

1. $y''' + 3y' + y = e^x - x - 1$

```
ClearAll["Global`*"]
```

First trying to solve the ODE.

```
dapple = y'''[x] + 3 y''[x] + 3 y'[x] + y[x] == ⅇ^x - x - 1
apple = DSolve[dapple, y[x], x]
```

$$y[x] + 3\,y'[x] + 3\,y''[x] + y^{(3)}[x] == -1 + e^x - x$$

$$\left\{\left\{y[x] \to \frac{1}{8}(16 + e^x - 8x) + e^{-x} C[1] + e^{-x} x\, C[2] + e^{-x} x^2 C[3]\right\}\right\}$$

I think I can improve the appearance a little.

```
Collect[apple, ⅇ^-x]
```

$$\left\{\left\{y[x] \to 2 + \frac{e^x}{8} - x + e^{-x}\left(C[1] + x\, C[2] + x^2 C[3]\right)\right\}\right\}$$

1. Above: The expression matches the answer in the text.

3. $(D^4 + 10 D^2 + 9 I)\, y = 6.5\, \text{Sinh}[2x]$

In[9]:=
```
ClearAll["Global`*"]
```

First trying to solve the ODE.

In[10]:=
```
prank = y''''[x] + 10 y''[x] + 9 y[x] == 6.5 Sinh[2 x]
dank = DSolve[prank, y[x], x]
```

Out[10]= $9\,y[x] + 10\,y''[x] + y^{(4)}[x] == 6.5\,\text{Sinh}[2x]$

Out[11]= $\{\{y[x] \to 1.\,C[3]\,\text{Cos}[1.\,x] + 1.\,C[1]\,\text{Cos}[3.\,x] + 1.\,C[4]\,\text{Sin}[1.\,x] + 1.\,C[2]\,\text{Sin}[3.\,x] + 0.1625\,(0. + 1.\,\text{Cos}[1.\,x]^2\,\text{Sinh}[2.\,x] - 0.384615\,\text{Cos}[3.\,x]^2\,\text{Sinh}[2.\,x] + 1.\,\text{Sin}[1.\,x]^2\,\text{Sinh}[2.\,x] - (0.384615 + 2.31296\times 10^{-17}\,i)\,\text{Sin}[3.\,x]^2\,\text{Sinh}[2.\,x])\}\}$

Then trying to eliminate the imaginary parts, which I think slipped in at the boundary of machine precision operations.

In[12]:=
```
bank = Chop[dank, 10^-16]
```

Out[12]=
```
{{y[x] → 1. C[3] Cos[1. x] +
    1. C[1] Cos[3. x] + 1. C[4] Sin[1. x] + 1. C[2] Sin[3. x] +
    0.1625 (1. Cos[1. x]^2 Sinh[2. x] - 0.384615 Cos[3. x]^2 Sinh[2. x] +
       1. Sin[1. x]^2 Sinh[2. x] - 0.384615 Sin[3. x]^2 Sinh[2. x])}}
```

And trying to compactify.

In[13]:=
```
sank = Simplify[bank]
```

Out[13]=
```
{{y[x] → 1. C[3] Cos[(1. + 0. i) x] + 1. C[1] Cos[(3. + 0. i) x] +
    1. C[4] Sin[(1. + 0. i) x] + 1. C[2] Sin[(3. + 0. i) x] + 0.1 Sinh[2. x]}}
```

And taking another shot at removing imaginaries.

In[14]:=
```
Chop[sank, 10^-16]
```

Out[14]=
```
{{y[x] → 1. C[3] Cos[1. x] + 1. C[1] Cos[3. x] +
    1. C[4] Sin[1. x] + 1. C[2] Sin[3. x] + 0.1 Sinh[2. x]}}
```

1. Above: The expression matches the text’s answer.

5. $(x^3 D^3 + x^2 D^2 - 2 x D + 2 I)\, y = x^{-2}$

```
ClearAll["Global`*"]
```

First trying to solve the ODE.

```
plow = x^3 y'''[x] + x^2 y''[x] - 2 x y'[x] + 2 y[x] == x^-2
cow = DSolve[plow, y[x], x]
```

$$2 y[x] - 2 x y'[x] + x^2 y''[x] + x^3 y^{(3)}[x] == \frac{1}{x^2}$$

$$\left\{\left\{y[x] \to -\frac{1}{12 x^2} + \frac{C[1]}{x} + x\, C[2] + x^2\, C[3]\right\}\right\}$$

1. Above: The answer matches the text’s.

7. $(D^3 - 9 D^2 + 27 D - 27 I)\, y = 27\, \text{Sin}[3 x]$

First trying to solve the ODE.

```
boat = y'''[x] - 9 y''[x] + 27 y'[x] - 27 y[x] == 27 Sin[3 x]
coat = DSolve[boat, y[x], x]
```

$$-27 y[x] + 27 y'[x] - 9 y''[x] + y^{(3)}[x] == 27\, \text{Sin}[3 x]$$

$$\left\{\left\{y[x] \to e^{3 x} C[1] + e^{3 x} x\, C[2] + e^{3 x} x^2\, C[3] + \frac{1}{4}(-\text{Cos}[3 x] + \text{Sin}[3 x])\right\}\right\}$$

And trying to introduce some organization.

```
goat = Collect[coat, e^(3 x)]
```

$$\left\{\left\{y[x] \to e^{3x}\left(C[1] + x\,C[2] + x^2\,C[3]\right) - \frac{1}{4}\mathrm{Cos}[3\,x] + \frac{1}{4}\mathrm{Sin}[3\,x]\right\}\right\}$$

1. Above: The answer matches the text's.

8 - 13 Initial value problem
Solve the given IVP.

9. $y^{iv} + 5\,y'' + 4\,y = 90\,\mathrm{Sin}[x],\ y[0] = 1,$
$y'[0] = 2,\ y''[0] = -1,\ y'''[0] = -32$

```
ClearAll["Global`*"]
```

First trying to solve the ODE.

```
sing = {y''''[x] + 5 y''[x] + 4 y[x] == 90 Sin[4 x],
  y[0] == 1, y'[0] == 2, y''[0] == -1, y'''[0] == -32}
ring = DSolve[sing, y[x], x]
```

$$\{4\,y[x] + 5\,y''[x] + y^{(4)}[x] == 90\,\mathrm{Sin}[4\,x],\ y[0] == 1,\ y'[0] == 2,\ y''[0] == -1,\ y^{(3)}[0] == -32\}$$

$$\Big\{\Big\{y[x] \to \frac{1}{4}\big(4\,\mathrm{Cos}[x] + 80\,\mathrm{Cos}[x]^3\,\mathrm{Sin}[x] - 40\,\mathrm{Cos}[3\,x]\,\mathrm{Sin}[x] - 12\,\mathrm{Cos}[5\,x]\,\mathrm{Sin}[x] - 80\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^3 + 15\,\mathrm{Cos}[2\,x]\,\mathrm{Sin}[2\,x] + 20\,\mathrm{Cos}[2\,x]^3\,\mathrm{Sin}[2\,x] - 40\,\mathrm{Cos}[x]\,\mathrm{Sin}[3\,x] + 12\,\mathrm{Cos}[x]\,\mathrm{Sin}[5\,x] - 5\,\mathrm{Cos}[2\,x]\,\mathrm{Sin}[6\,x]\big)\Big\}\Big\}$$

Below I do some hammering to try to get the Mathematica solution into the same form as the text answer.

```
thing = Simplify[ring]
```

{{y[x] → Cos[x] (1 - Sin[x] + Sin[3 x])}}

1. Below: To see what I need to make equal to $\frac{1}{2}\mathrm{Sin}[4\,x]$.

```
TrigExpand[-Cos[x] Sin[x] + Cos[x] Sin[3 x]]
```

$$2\,\mathrm{Cos}[x]^3\,\mathrm{Sin}[x] - 2\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^3$$

```
bling = thing /.
  (Cos[x] (1 - Sin[x] + Sin[3 x])) → (Cos[x] - Cos[x] Sin[x] + Cos[x] Sin[3 x])
```

{{y[x] → Cos[x] - Cos[x] Sin[x] + Cos[x] Sin[3 x]}}

---

2. Below: Putting together some idents to use.

```
TrigExpand[Sin[2 x]]
2 Cos[x] Sin[x]

TrigExpand[Sin[3 x]]
3 Cos[x]^2 Sin[x] - Sin[x]^3

TrigExpand[Cos[2 x]]
Cos[x]^2 - Sin[x]^2
```

3. Therefore $\text{Sin}[4x] == 2\,\text{Cos}[2x]\,\text{Sin}[2x] == 2\left(\left(\text{Cos}[x]^2 - \text{Sin}[x]^2\right)(2\,\text{Cos}[x]\,\text{Sin}[x])\right)$

The following five substitution attempts do not condense very much

```
sling = bling /. (Sin[3 x]) → (3 Cos[x]^2 Sin[x] - Sin[x]^3)
{{y[x] → Cos[x] - Cos[x] Sin[x] + Cos[x] (3 Cos[x]^2 Sin[x] - Sin[x]^3)}}

string = sling /. (Cos[x] (3 Cos[x]^2 Sin[x] - Sin[x]^3)) ->
    (Cos[x] Sin[x] (3 Cos[x]^2 - Sin[x]^2))
{{y[x] → Cos[x] - Cos[x] Sin[x] + Cos[x] Sin[x] (3 Cos[x]^2 - Sin[x]^2)}}

zing = string /. (3 Cos[x]^2 - Sin[x]^2) → ( 2 Cos[x]^2 + Cos[2 x])
{{y[x] → Cos[x] - Cos[x] Sin[x] + Cos[x] (2 Cos[x]^2 + Cos[2 x]) Sin[x]}}

fling = zing /.
   (Cos[x] (2 Cos[x]^2 + Cos[2 x]) Sin[x]) → (1/2 Sin[2 x] (Cos[2 x] + 2 Cos[x]^2))
{{y[x] → Cos[x] - Cos[x] Sin[x] + 1/2 (2 Cos[x]^2 + Cos[2 x]) Sin[2 x]}}

ping = fling /.
   (1/2 (2 Cos[x]^2 + Cos[2 x]) Sin[2 x]) → (1/2 (2 Cos[x]^2 Sin[2 x] + 1/2 Sin[4 x]))
{{y[x] → Cos[x] - Cos[x] Sin[x] + 1/2 (2 Cos[x]^2 Sin[2 x] + 1/2 Sin[4 x])}}
```

4. So I decide it's time to swing for the fence. I sequester one factor of Cos[x], simplify the rest, then reassemble.

```
p1 = Cos[x]
Cos[x]
```

```
p2 = Simplify[-Cos[x] Sin[x] + 1/2 (2 Cos[x]^2 Sin[2 x] + 1/2 Sin[4 x])]
```

$\frac{1}{2}$ Sin[4 x]

5. So that I can write.

```
out = p1 + p2
```

Cos[x] + $\frac{1}{2}$ Sin[4 x]

6. Above: The answer does match the text answer.

11. $(D^3 - 2 D^2 - 3 D)$ y = 74 e^{-3x} Sin[x],
y[0] = -1.4, y'[0] = 3.2, y''[0] = -5.2

```
ClearAll["Global`*"]
```

First trying to solve the ODE.

```
alt = {y'''[x] - 2 y''[x] - 3 y'[x] == 74 e^(-3 x) Sin[x],
  y[0] == -1.4, y'[0] == 3.2, y''[0] == -5.2}
kalt = DSolve[alt, y[x], x]
```

{-3 y′[x] - 2 y″[x] + $y^{(3)}$[x] == 74 e^{-3x} Sin[x],
y[0] == -1.4, y′[0] == 3.2, y″[0] == -5.2}

{{y[x] → -$\frac{1}{5}$ e^{-3x} (7 Cos[x] + 5 Sin[x])}}

Followed by a presumptuous but possibly amusing wholesale substitution as a mean of recasting

```
salt = kalt /. (-1/5 e^(-3 x) (7 Cos[x] + 5 Sin[x])) → (e^(-3 x) (-7/5 Cos[x] - 5/5 Sin[x]))
```

{{y[x] → e^{-3x} (-$\frac{7 \text{Cos}[x]}{5}$ - Sin[x])}}

1. Above: Substitution by hand results in the text's answer.

13. $(D^3 - 4 D)$ y = 10 Cos[x] + 5 Sin[x], y[0] = 3, y'[0] = -2, y''[0] = -1

```
ClearAll["Global`*"]
```

```
rog = {y'''[x] - 4 y'[x] == 10 Cos[x] + 5 Sin[x],
  y[0] == 3, y'[0] == -2, y''[0] == -1}
dog = DSolve[rog, y[x], x]
```

$\{-4\, y'[x] + y^{(3)}[x] == 10\, \mathrm{Cos}[x] + 5\, \mathrm{Sin}[x],\ y[0] == 3,\ y'[0] == -2,\ y''[0] == -1\}$

```
{{y[x] → 2 + Cos[x] - 2 Sin[x]}}
```

1. Above: The answer matches the text’s.